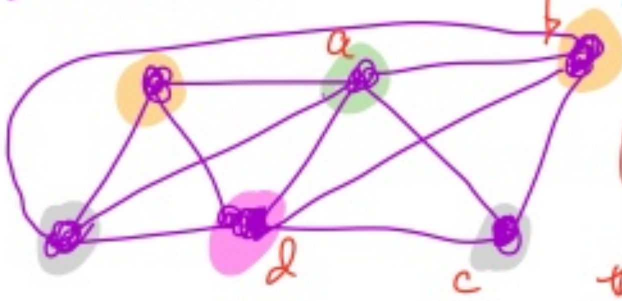
Question: I have to color the vertices of this graph below so that the same color on 2 vertices cannot be connected by an edge. What is the smallest # of different colors?



This shows minimum ≤ 4

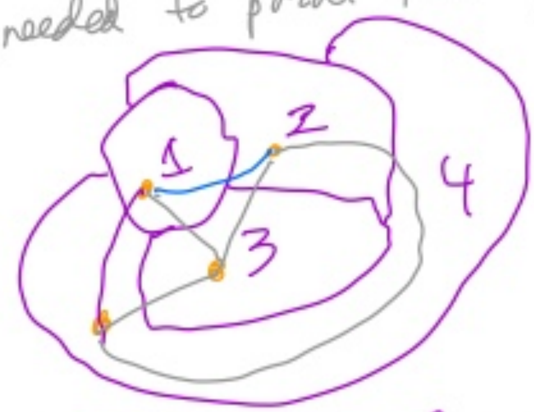
Answer: 4

We can show ≤ 3 colors is impossible. The part of the graph with $\{a, b, c, d\}$ is K_4 — all vertices are connected to the other 3 → All 4 must have distinct colors.

The minimum # of colors is called the chromatic number of the graph.

Note: By similar reasoning, the chromatic # of K_n is n.

This is related to a big math question that was only solved in the 1980's — the map coloring problem. The question — given a map with different countries (regions that are topologically disks) — what is the fewest # of colors needed to print the map?



Need at least 4 colors.

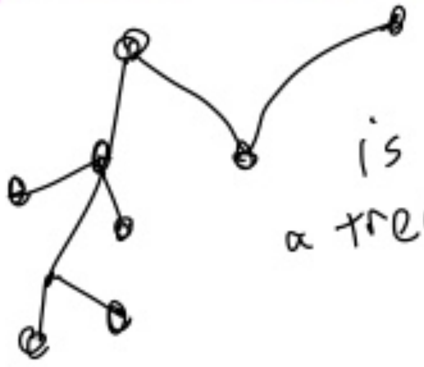
Thm (1980's) — 4 is enough (4-color map Theorem).

To solve it, the researchers turned it into a planar graph theory problem ← each country/region is a vertex, vertices are connected with an edge if there is a common boundary between them.

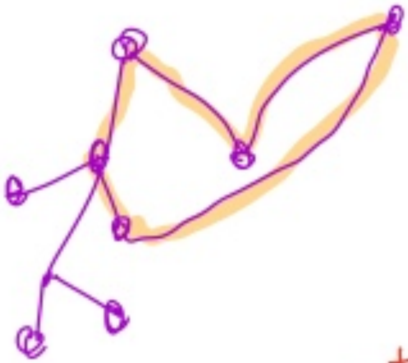
— Then the question becomes a question about planar graphs. — They figured out how to reduce the problem to solving 1936 cases ← they checked them all with a computer. People have tried to solve it in a simpler way — some have reduced the # of cases to check to 600. This is the first example of a math proof aided by a computer. So far, no one knows a simpler proof.

Trees

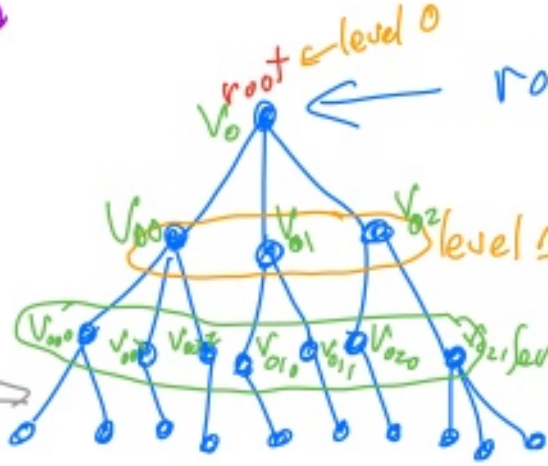
A tree is a simple connected undirected graph that contains no simple circuits.



is a tree



is not a tree (has simple circuit of length 5).



rooted tree: one vertex is designated as the root

level 1 – shortest path to root is length 1.

level 2 – shortest path to root is length 2.

V_{00} and V_{01} and V_{02} are <u>children</u> of V_0

V_0 is the <u>parent of</u> V_{00}, V_{01}, V_{02}

V_{00} and V_{01} are <u>siblings</u>

V_{001} is a <u>descendant</u> of V_0

These edges are called <u>leaves</u> of the tree.
A <u>leaf</u> is an edge on the tree such that one of its boundary vertices has degree 1.

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binary tree
||
a rooted tree where every vertex has at most 2 children

full binary tree
||
a rooted tree where every vertex has exactly 2 children.

Generalizations: ternary

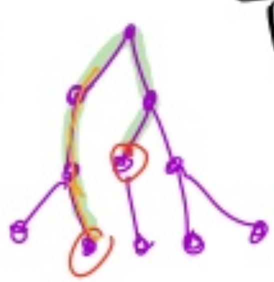
m-ary tree

a rooted tree where each vertex has at most m children

a full m-ary tree is a rooted tree where every vertex has exactly m children.

Theorem An undirected simple graph is a tree $\Longleftrightarrow$ there exists a unique simple path between any two vertices.

Example



idea of Proof: Given a tree, there is a simple path between any two vertices. If there were two different ones, we would be able to make a simple circuit.

Fact Given a tree with $v = n$ vertices, there are $(n-1)$ edges.

Proof: stay tuned...